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The Accurate Inversion of Vandermonde Matrices

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Abstract—Two modifications are suggested in the commonly used algorithms (such as the $O(n^2)$ Parker algorithm) for the explicit inversion of Vandermonde matrices resulting in an algorithm whose accuracy is no worse than those of the existing algorithms, but which is significantly more accurate in many pathological situations. The first modification circumvents, to some extent, the subtraction of 'two big like-signed numbers' which in turn reduces round-off errors, while the second modification exploits the structure of the inverse and uses two recursive formulae instead of one to bring about an increase in accuracy. Numerical results are presented to demonstrate the increase in accuracy that results from these two modifications. Although the modified algorithm is always at least as accurate as the Parker algorithm, it does, unfortunately, involve an increase in complexity from $O(n^2)$ to $O(n^3)$, so that use of this algorithm to increase the relative accuracy is recommended only in situations where the standard algorithms fail to yield accurate results.

Keywords—Accurate Vandermonde inverse.

1. INTRODUCTION

There is a large amount of literature on the computation of the analytical inverse of Vandermonde matrices; some of the early algorithms are described in [1–4]. Gohberg and Olshevsky [5] in their work compare the performance of some of the well-known algorithms. They note that, in general, standard numerical schemes fail to compute the inverse of a Vandermonde matrix accurately. Thus, it is necessary to exploit the known structure of a Vandermonde matrix in order to achieve high relative accuracy in computing the inverse. They also note that the Parker algorithm of [1] differs from the Traub algorithm of [3] in only one nonessential detail; however, this small difference is critical in obtaining higher numerical accuracy. Their comparison of the Parker algorithm with the Bjorck-Pereyra algorithm of [4] revealed that even in a situation which was most favorable for the Bjorck-Pereyra algorithm, the performance of the Parker algorithm was not worse, and in other situations it was much better. Thus, their conclusion is that the Parker algorithm is not only accurate but also fast with a complexity of $6n^2$ floating point operations.

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Jog [6] suggested a new method for computing the explicit inverse of a Vandermonde matrix based on the Newton identities. However, the implementation details and a comparison with existing algorithms such as the Parker algorithm, being outside the scope of that article, were not provided. It is the purpose of this article to provide these details. Since, from the above discussion, it is clear that the Parker algorithm is not only fast but also accurate, we shall compare our algorithm with the Parker algorithm alone.

2. DETAILS OF THE PROPOSED ALGORITHM

We are interested in numerically computing, as *accurately* as possible, the inverse of the $n \times n$ Vandermonde matrix given by

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}. \tag{1}$$

Let $\mathbf{R}_i$ denote the diagonal matrix with all the λ_j, $j = 1, \ldots, n$, $j \neq i$, along the diagonal, i.e., $\mathbf{R}_1 = \operatorname{diag}[\lambda_2, \lambda_3, \ldots, \lambda_n]$, $\mathbf{R}_2 = \operatorname{diag}[\lambda_1, \lambda_3, \ldots, \lambda_n]$, and so on. Then, as noted in [6], the denominator of the i^{th} row of the inverse of the matrix in equation (1) is given by

$$D_i = \prod_{\substack{j \neq i \\ j=1}}^{n} (\lambda_i - \lambda_j),$$

while the numerators in row i are given by $\{(-1)^{n-1}(I_{n-1})_{\mathbf{R}_i}, (-1)^{n-2}(I_{n-2})_{\mathbf{R}_i}, \ldots, 1\}$, where $\{(I_1)_{\mathbf{R}_i}, (I_2)_{\mathbf{R}_i}, \ldots\}$ denote the principal invariants of $\mathbf{R}_i$. The fact that the terms in the numerator of the i^{th} row are the principal invariants of the diagonal matrix $\mathbf{R}_i$ immediately suggests the use of the following Newton identities for their evaluation (see [7] and references therein for various proofs of the Newton identities):

$$I_k^{(i)} = \frac{(-1)^{k+1}}{k} \left[\operatorname{tr} \mathbf{R}_i^k - I_1 \operatorname{tr} \mathbf{R}_i^{k-1} + \cdots + (-1)^{k-1} I_{k-1} \operatorname{tr} \mathbf{R}_i \right],$$

where $I_k^{(i)} \equiv (I_k)_{\mathbf{R}_i}$. In our numerical implementation, the I_k for each row i are found recursively using the relation

$$I_k^{(i)} = \frac{(-1)^{k+1}}{k} \operatorname{tr} \left(\mathbf{H}_{k-1}^{(i)} \mathbf{R}_i \right); \qquad k = 1, \ldots, n, \tag{2}$$

where $\mathbf{H}_0 \equiv \mathbf{I}$, and

$$\mathbf{H}_{k-1}^{(i)} = \mathbf{H}_{k-2}^{(i)} \mathbf{R}_i + (-1)^{k-1} I_{k-1}^{(i)} \mathbf{I}; \qquad k = 2, \ldots, n.$$

Note that the task of finding the product $\mathbf{H}_{k-1}^{(i)} \mathbf{R}_i$ for each k is a trivial one, since $\mathbf{H}_{k-1}^{(i)}$ and $\mathbf{R}_i$ are diagonal matrices.

To see the way the current algorithm differs from the Parker algorithm, note that the use of equation (2) to compute the entries of the i^{th} row *does not* involve λ_i, since each $\mathbf{R}_i$ is a diagonal matrix with λ_i excluded. This increases the complexity of the algorithm from $O(n^2)$ to $O(n^3)$. However, it is precisely this feature that prevents, to some extent, the subtraction of two big 'like-signed' numbers as we now show, and increases the accuracy of the method. Similar to the Parker algorithm, we can get an $O(n^2)$ complexity algorithm by first computing the principal invariants of the diagonal matrix with *all* λ_i along the diagonal, i.e., computing the principal invariants, I_i, $i = 1, \ldots, n$, of $\mathrm{diag}[\lambda_1, \ldots, \lambda_n]$, and then computing the numerators of each row i, using the recursion

$$I_j^{(i)} = I_j - \lambda_i I_{j-1}^{(i)}; \qquad j = 1, \ldots, n-1, \quad I_0^{(i)} \equiv 1.$$

However, when λ_i is a positive number, using the above formula can cause subtraction of two big 'like-signed' numbers, just as in the case of the Parker algorithm, resulting in a propagation of round-off errors.

A further increase in accuracy, in some situations, can be obtained by the use of the conjugate Newton identity that makes it possible to find the lower-order invariants from the higher-order ones, and which is given by [6]

$$I_{n-j}^{(i)} = \frac{(-1)^{j+1}}{j} \operatorname{tr}\left[I_n \mathbf{R}_i^{-j} - I_{n-1}\mathbf{R}_i^{-j+1} + \cdots + (-1)^{j-1} I_{n-j+1}\mathbf{R}_i^{-1}\right]; \qquad j = 1, \ldots, n-1.$$

Of course, the above identity requires that each $\mathbf{R}_i$ should be invertible, so that, if one of the λ_i is zero, then it cannot be used. In such a situation (which is, in any case, rare in practical applications), we simply use the Newton, instead of the conjugate Newton, identity to compute the numerators.

To the best of our knowledge, the conjugate Newton identity does not seem to have been documented in the literature (prior to [6]). Since it involves the inverse of a matrix and its powers, it is, in general, cumbersome to use. However, in our situation, computing each $\mathbf{R}_i^{-1}$ and its powers is trivial, since $\mathbf{R}_i$ is a diagonal matrix; in fact, the computational complexity of using the conjugate Newton identity is the same as using the Newton identity itself. To see why the use of the conjugate Newton identity leads to an increase in relative accuracy, note that round-off errors tend to propagate if we compute all the invariants in the numerators of the inverse using the Newton identity alone. The structure of the inverse shows that, not only the first invariant of $\mathbf{R}_i$, but also the last invariant involved in the inverse, namely $(I_{n-1})_{\mathbf{R}_i}$, is easy to compute directly using *only the input data*, since it is simply the product of the elements of $\mathbf{R}_i$. Thus, computing the first $n - m$ columns of the matrix using the conjugate Newton identity, and the remaining m columns using the Newton identity itself, where m is appropriately chosen, brings about a relative increase in accuracy. We have found by means of numerical experiments that values of (approximately) $m = 2n/3$ and $m = n/3$ seem to work well when the λ_i are monotonically increasing and decreasing, respectively, although, of course, since m is an input to our program, one can try out various choices to find out the value that works best.

Summarizing, the two changes suggested to the Parker algorithm to bring about a relative increase in accuracy are the following.

IM1 Do not involve λ_i in the computation of the numerators of row i of the inverse. This modification increases the complexity of the algorithm from $O(n^2)$ to $O(n^3)$.

IM2 Since the first and last columns of the inverse can be computed directly, use two recursive formulae, one of which proceeds recursively from the first column to some appropriately chosen intermediate column, and the other from the last column up to this intermediate column. This modification does not increase the complexity of the algorithm.

We emphasize at this stage that the above two changes do not guarantee an increase in accuracy relative to the Parker algorithm in *all* cases (see Example 5 in the following section), although, of course, the accuracy is not worse either. However, when the errors are of the type addressed by the two changes, there is a significant increase in accuracy as the examples presented in the following section demonstrate.

3. NUMERICAL RESULTS

We shall present five examples, where the λ_i occur in different sequences, such as positive and monotonically increasing, monotonically increasing in magnitude but with alternating signs, etc.

We shall first present the results given by the Parker algorithm, then by the modified algorithm involving step IM1 alone (i.e., with $m = n$) and, finally, by the modified algorithm involving both modifications IM1 and IM2 (typically by using $m \approx 2n/3$). In this way, the successive increase in accuracy obtained over the Parker algorithm due to the two modifications will become evident. Of course, this is done merely to show the relative contributions of IM1 and IM2; in practice, one would have to execute the algorithm only once with *both* IM1 and IM2 operative. In what follows, the Vandermonde matrix and its inverse are denoted by V_M and V_M^{-1}, respectively. We prefer to present V_M^{-1} and $V_M V_M^{-1}$ in their entirety instead of merely presenting a norm measure of the error, so that the exact location of the dominant errors and the improvement due to the suggested modifications is immediately evident.[1] All computations are carried out in double precision (real$\star$8) unless stated otherwise. Computing $V_M V_M^{-1}$ involves multiplying two matrices and may result in round-off errors. Thus, although the inverse V_M^{-1} may be computed accurately, the product $V_M V_M^{-1}$ may display large errors (see Examples 4 and 5 where this occurs). Hence, in addition to carrying out the backward accuracy check of computing $V_M V_M^{-1}$, we also carry out a forward accuracy check by computing the inverse by our strategy using real$\star$16 precision, and taking this as the 'exact' inverse for the purpose of comparison. In all the examples, we find that the inverses computed using the real$\star$8 and real$\star$16 precisions differ only in the 16^{th} significant digit, thus showing the high accuracy to which the inverse is being computed by the proposed method. Since the first 15 significant digits computed with the real$\star$8 and real$\star$16 precisions are identical, we shall only present the results for $V_M V_M^{-1}$ in the real$\star$16 case, which incidentally are very close to $\mathbf{I}$, showing that the results for V_M^{-1} obtained using real$\star$16 precision can indeed be taken as the 'exact' solution.

EXAMPLE 1. Let $n = 6$, and let $\lambda_1 = 1$, $\lambda_2 = 3\text{E}1$, $\lambda_3 = 9\text{E}2$, $\lambda_4 = 81\text{E}3$, $\lambda_5 = 243\text{E}4$, $\lambda_6 = 729\text{E}5$.

PARKER ALGORITHM.

$$V_M^{-1} = \begin{bmatrix} 0.10356\text{E}{+}01 & -0.35686\text{E}{-}01 & 0.38813\text{E}{-}04 & -0.49005\text{E}{-}09 & 0.20159\text{E}{-}15 & -0.26732\text{E}{-}23 \\ -0.35685\text{E}{-}01 & 0.35726\text{E}{-}01 & -0.40107\text{E}{-}04 & 0.50656\text{E}{-}09 & -0.20839\text{E}{-}15 & 0.27633\text{E}{-}23 \\ 0.38803\text{E}{-}04 & -0.40097\text{E}{-}04 & 0.12939\text{E}{-}05 & -0.16518\text{E}{-}10 & 0.67976\text{E}{-}17 & -0.90141\text{E}{-}25 \\ -0.53227\text{E}{-}10 & 0.55061\text{E}{-}10 & -0.18354\text{E}{-}11 & 0.19722\text{E}{-}14 & -0.83832\text{E}{-}21 & 0.11128\text{E}{-}28 \\ 0.65458\text{E}{-}16 & -0.69462\text{E}{-}16 & 0.23162\text{E}{-}17 & -0.25156\text{E}{-}20 & 0.30738\text{E}{-}25 & -0.42117\text{E}{-}33 \\ 0.18151\text{E}{-}20 & 0.27653\text{E}{-}23 & -0.92211\text{E}{-}25 & 0.10018\text{E}{-}27 & -0.12635\text{E}{-}32 & 0.50301\text{E}{-}39 \end{bmatrix},$$

$$V_M V_M^{-1} = \begin{bmatrix} 0.10000\text{E}{+}01 & 0.21600\text{E}{-}17 & 0.18554\text{E}{-}20 & 0.73140\text{E}{-}25 & 0.49924\text{E}{-}32 & 0.16094\text{E}{-}39 \\ -0.50473\text{E}{-}13 & 0.10000\text{E}{+}01 & 0.15997\text{E}{-}18 & 0.21549\text{E}{-}23 & 0.10539\text{E}{-}29 & 0.42062\text{E}{-}38 \\ -0.76203\text{E}{-}08 & -0.48099\text{E}{-}13 & 0.10000\text{E}{+}01 & -0.83041\text{E}{-}21 & 0.48590\text{E}{-}27 & 0.46974\text{E}{-}36 \\ 0.68000\text{E}{+}03 & -0.76226\text{E}{-}08 & 0.53739\text{E}{-}13 & 0.10000\text{E}{+}01 & 0.52264\text{E}{-}22 & -0.63772\text{E}{-}30 \\ 0.51280\text{E}{+}11 & 0.68000\text{E}{+}03 & 0.58008\text{E}{-}08 & -0.37685\text{E}{-}11 & 0.10000\text{E}{+}01 & -0.18644\text{E}{-}23 \\ 0.37424\text{E}{+}19 & 0.51280\text{E}{+}11 & 0.67999\text{E}{+}03 & 0.31516\text{E}{-}04 & 0.40700\text{E}{-}10 & 0.10000\text{E}{+}01 \end{bmatrix}.$$

MODIFIED ALGORITHM WITH IM1 ALONE ($m = 6$).

$$V_M^{-1} = \begin{bmatrix} 0.10356\text{E}{+}01 & -0.35686\text{E}{-}01 & 0.38813\text{E}{-}04 & -0.49005\text{E}{-}09 & 0.20159\text{E}{-}15 & -0.26732\text{E}{-}23 \\ -0.35685\text{E}{-}01 & 0.35726\text{E}{-}01 & -0.40107\text{E}{-}04 & 0.50656\text{E}{-}09 & -0.20839\text{E}{-}15 & 0.27633\text{E}{-}23 \\ 0.38771\text{E}{-}04 & -0.40097\text{E}{-}04 & 0.12939\text{E}{-}05 & -0.16518\text{E}{-}10 & 0.67976\text{E}{-}17 & -0.90141\text{E}{-}25 \\ -0.53158\text{E}{-}10 & 0.55061\text{E}{-}10 & -0.18354\text{E}{-}11 & 0.19722\text{E}{-}14 & -0.83832\text{E}{-}21 & 0.11128\text{E}{-}28 \\ 0.67149\text{E}{-}16 & -0.69462\text{E}{-}16 & 0.23162\text{E}{-}17 & -0.25156\text{E}{-}20 & 0.30738\text{E}{-}25 & -0.42117\text{E}{-}33 \\ -0.26732\text{E}{-}23 & 0.27653\text{E}{-}23 & -0.92211\text{E}{-}25 & 0.10018\text{E}{-}27 & -0.12635\text{E}{-}32 & 0.50301\text{E}{-}39 \end{bmatrix},$$

$$V_M V_M^{-1} = \begin{bmatrix} 0.10000\text{E}{+}01 & -0.22399\text{E}{-}13 & 0.18554\text{E}{-}20 & 0.73140\text{E}{-}25 & 0.49924\text{E}{-}32 & 0.16094\text{E}{-}39 \\ -0.26881\text{E}{-}04 & 0.10000\text{E}{+}01 & 0.15935\text{E}{-}18 & 0.21549\text{E}{-}23 & 0.10539\text{E}{-}29 & 0.42062\text{E}{-}38 \\ -0.24921\text{E}{-}01 & -0.51808\text{E}{-}09 & 0.10000\text{E}{+}01 & -0.83041\text{E}{-}21 & 0.48590\text{E}{-}27 & 0.46974\text{E}{-}36 \\ 0.14021\text{E}{+}02 & 0.41404\text{E}{-}06 & -0.45209\text{E}{-}13 & 0.10000\text{E}{+}01 & 0.52264\text{E}{-}22 & -0.63772\text{E}{-}30 \\ 0.29699\text{E}{+}07 & 0.71743\text{E}{-}01 & 0.95542\text{E}{-}08 & -0.37685\text{E}{-}11 & 0.10000\text{E}{+}01 & -0.18644\text{E}{-}23 \\ 0.24221\text{E}{+}12 & 0.58454\text{E}{+}04 & -0.19897\text{E}{-}01 & 0.31516\text{E}{-}04 & 0.40700\text{E}{-}10 & 0.10000\text{E}{+}01 \end{bmatrix}.$$

[1] Since we present only the first five significant digits of the entries of V_M^{-1} and $V_M V_M^{-1}$ due to space constraints, in some cases, the improvement in the computed inverse, when it occurs beyond the first five digits, will be evident from $V_M V_M^{-1}$ and not from V_M^{-1}.

Modified Algorithm with Both, IM1 and IM2 ($m = 4$).

$$V_M^{-1} = \begin{bmatrix} 0.10356E{+}01 & -0.35686E{-}01 & 0.38813E{-}04 & -0.49005E{-}09 & 0.20159E{-}15 & -0.26732E{-}23 \\ -0.35685E{-}01 & 0.35726E{-}01 & -0.40107E{-}04 & 0.50656E{-}09 & -0.20839E{-}15 & 0.27633E{-}23 \\ 0.38803E{-}04 & -0.40097E{-}04 & 0.12939E{-}05 & -0.16518E{-}10 & 0.67976E{-}17 & -0.90141E{-}25 \\ -0.53227E{-}10 & 0.55061E{-}10 & -0.18354E{-}11 & 0.19722E{-}14 & -0.83832E{-}21 & 0.11128E{-}28 \\ 0.67149E{-}16 & -0.69462E{-}16 & 0.23162E{-}17 & -0.25156E{-}20 & 0.30738E{-}25 & -0.42117E{-}33 \\ -0.26732E{-}23 & 0.27653E{-}23 & -0.92211E{-}25 & 0.10018E{-}27 & -0.12635E{-}32 & 0.50301E{-}39 \end{bmatrix},$$

$$V_M V_M^{-1} = \begin{bmatrix} 0.10000E{+}01 & -0.48473E{-}17 & 0.18554E{-}20 & 0.73140E{-}25 & 0.49924E{-}32 & 0.16094E{-}39 \\ 0.69584E{-}16 & 0.10000E{+}01 & 0.15935E{-}18 & 0.21549E{-}23 & 0.10539E{-}29 & 0.42062E{-}38 \\ 0.14263E{-}14 & -0.10137E{-}13 & 0.10000E{+}01 & -0.83041E{-}21 & 0.48590E{-}27 & 0.46974E{-}36 \\ 0.29482E{-}11 & -0.41826E{-}11 & -0.45209E{-}13 & 0.10000E{+}01 & 0.52264E{-}22 & -0.63772E{-}30 \\ 0.19668E{-}06 & -0.34746E{-}06 & 0.95542E{-}08 & -0.37685E{-}11 & 0.10000E{+}01 & -0.18644E{-}23 \\ 0.19971E{+}00 & 0.37842E{+}00 & -0.19897E{-}01 & 0.31516E{-}04 & 0.40700E{-}10 & 0.10000E{+}01 \end{bmatrix}.$$

The significant improvement in accuracy in the computed inverse by incorporating both IM1 and IM2 is evident.

Modified Algorithm with Both, IM1 and IM2 ($m = 4$, precision: real$\star$16).

$$V_M V_M^{-1} = \begin{bmatrix} 0.10000E{+}01 & 0.11678E{-}32 & -0.60616E{-}37 & 0.34121E{-}41 & -0.26451E{-}47 & -0.21993E{-}55 \\ 0.52198E{-}33 & 0.10000E{+}01 & 0.19646E{-}35 & 0.67965E{-}40 & 0.53982E{-}46 & -0.30623E{-}54 \\ -0.69273E{-}31 & -0.57393E{-}31 & 0.10000E{+}01 & -0.23911E{-}36 & 0.10606E{-}42 & -0.94731E{-}51 \\ -0.46567E{-}28 & 0.12215E{-}28 & 0.20832E{-}28 & 0.10000E{+}01 & 0.41211E{-}38 & -0.10811E{-}45 \\ 0.24815E{-}23 & 0.23988E{-}22 & 0.32312E{-}23 & -0.10855E{-}26 & 0.10000E{+}01 & -0.36734E{-}39 \\ 0.16653E{-}15 & -0.16653E{-}15 & 0.17347E{-}17 & -0.33881E{-}20 & 0.10340E{-}24 & 0.10000E{+}01 \end{bmatrix}.$$

Example 2. Let $n = 7$, and let $\lambda_i = -(10)^{n-i}$, $i = 1, \ldots, 7$. Although this example is similar to the first example in that the λ_i are monotonically arranged, it is different in that the signs of the rows of V_M alternate.

Parker Algorithm.

$$V_M^{-1} =$$

$$\begin{bmatrix} 0.11955E{-}20 & 0.12484E{-}20 & 0.12610E{-}21 & 0.12622E{-}23 & 0.12610E{-}26 & 0.12484E{-}30 & 0.11236E{-}35 \\ -0.12485E{-}14 & -0.13871E{-}14 & -0.14010E{-}15 & -0.14011E{-}17 & -0.13885E{-}20 & -0.12623E{-}24 & -0.12484E{-}30 \\ 0.12610E{-}09 & 0.14010E{-}09 & 0.14139E{-}10 & 0.14026E{-}12 & 0.12764E{-}15 & 0.13885E{-}20 & 0.12610E{-}26 \\ -0.12622E{-}05 & -0.14011E{-}05 & -0.14026E{-}06 & -0.12777E{-}08 & -0.14026E{-}12 & -0.14011E{-}17 & -0.12622E{-}23 \\ 0.12610E{-}02 & 0.13885E{-}02 & 0.12764E{-}03 & 0.14026E{-}06 & 0.14139E{-}10 & 0.14010E{-}15 & 0.12610E{-}21 \\ -0.12484E{+}00 & -0.12623E{+}00 & -0.13885E{-}02 & -0.14011E{-}05 & -0.14010E{-}09 & -0.13871E{-}14 & -0.12484E{-}20 \\ 0.11236E{+}01 & 0.12484E{+}00 & 0.12610E{-}02 & 0.12622E{-}05 & 0.12610E{-}09 & 0.12484E{-}14 & 0.11236E{-}20 \end{bmatrix},$$

$$V_M V_M^{-1} =$$

$$\begin{bmatrix} 0.10000E{+}01 & 0.00000E{+}00 & 0.21684E{-}18 & 0.21176E{-}21 & 0.00000E{+}00 & 0.19722E{-}30 & 0.00000E{+}00 \\ 0.73275E{-}14 & 0.10000E{+}01 & -0.30358E{-}17 & -0.31764E{-}20 & -0.15510E{-}24 & -0.39443E{-}30 & -0.56424E{-}36 \\ -0.68257E{-}12 & 0.78548E{-}14 & 0.10000E{+}01 & 0.15734E{-}18 & 0.46012E{-}23 & 0.33921E{-}28 & 0.89338E{-}34 \\ 0.67917E{-}10 & -0.56119E{-}12 & 0.10378E{-}13 & 0.10000E{+}01 & -0.72381E{-}20 & -0.13531E{-}24 & -0.17467E{-}30 \\ 0.64000E{+}02 & 0.84642E{-}11 & -0.12527E{-}10 & 0.96021E{-}13 & 0.10000E{+}01 & 0.31471E{-}21 & 0.46481E{-}27 \\ -0.71106E{+}08 & 0.64000E{+}02 & 0.32940E{-}06 & 0.26390E{-}08 & 0.15571E{-}12 & 0.10000E{+}01 & -0.69004E{-}23 \\ 0.71824E{+}14 & -0.71106E{+}08 & 0.64000E{+}02 & -0.15871E{-}03 & 0.43217E{-}07 & -0.56032E{-}11 & 0.10000E{+}01 \end{bmatrix}.$$

Modified Algorithm with IM1 Alone ($m = 7$).

$$V_M^{-1} =$$

$$\begin{bmatrix} 0.11236E{-}20 & 0.12484E{-}20 & 0.12610E{-}21 & 0.12622E{-}23 & 0.12610E{-}26 & 0.12484E{-}30 & 0.11236E{-}35 \\ -0.12484E{-}14 & -0.13871E{-}14 & -0.14010E{-}15 & -0.14011E{-}17 & -0.13885E{-}20 & -0.12623E{-}24 & -0.12484E{-}30 \\ 0.12610E{-}09 & 0.14010E{-}09 & 0.14139E{-}10 & 0.14026E{-}12 & 0.12764E{-}15 & 0.13885E{-}20 & 0.12610E{-}26 \\ -0.12622E{-}05 & -0.14011E{-}05 & -0.14026E{-}06 & -0.12777E{-}08 & -0.14026E{-}12 & -0.14011E{-}17 & -0.12622E{-}23 \\ 0.12610E{-}02 & 0.13885E{-}02 & 0.12764E{-}03 & 0.14026E{-}06 & 0.14139E{-}10 & 0.14010E{-}15 & 0.12610E{-}21 \\ -0.12484E{+}00 & -0.12623E{+}00 & -0.13885E{-}02 & -0.14011E{-}05 & -0.14010E{-}09 & -0.13871E{-}14 & -0.12484E{-}20 \\ 0.11236E{+}01 & 0.12484E{+}00 & 0.12610E{-}02 & 0.12622E{-}05 & 0.12610E{-}09 & 0.12484E{-}14 & 0.11236E{-}20 \end{bmatrix},$$

$$V_M V_M^{-1} =$$

$$\begin{bmatrix} 0.10000E{+}01 & 0.71748E{-}14 & 0.21684E{-}18 & 0.21176E{-}21 & 0.00000E{+}00 & 0.19722E{-}30 & 0.00000E{+}00 \\ 0.41069E{-}06 & 0.10000E{+}01 & -0.47705E{-}17 & -0.31764E{-}20 & -0.15510E{-}24 & -0.39443E{-}30 & -0.56424E{-}36 \\ -0.41065E{-}04 & 0.71732E{-}10 & 0.10000E{+}01 & 0.15734E{-}18 & 0.46012E{-}23 & 0.33921E{-}28 & 0.89338E{-}34 \\ 0.41065E{-}02 & -0.71734E{-}08 & -0.59344E{-}13 & 0.10000E{+}01 & -0.72381E{-}20 & -0.13531E{-}24 & -0.17467E{-}30 \\ -0.41065E{+}00 & 0.71765E{-}06 & 0.24745E{-}10 & 0.96021E{-}13 & 0.10000E{+}01 & 0.31471E{-}21 & 0.46481E{-}27 \\ 0.41065E{+}02 & -0.72117E{-}04 & 0.43204E{-}06 & 0.26390E{-}08 & 0.15571E{-}12 & 0.10000E{+}01 & -0.69004E{-}23 \\ -0.41065E{+}04 & -0.10728E{+}00 & -0.16108E{-}01 & -0.15871E{-}03 & 0.43217E{-}07 & -0.56032E{-}11 & 0.10000E{+}01 \end{bmatrix}.$$

MODIFIED ALGORITHM WITH IM1 AND IM2 ($m = 5$).

$$V_M^{-1} =$$
$$\begin{bmatrix}
0.11236E-20 & 0.12484E-20 & 0.12610E-21 & 0.12622E-23 & 0.12610E-26 & 0.12484E-30 & 0.11236E-35 \\
-0.12484E-14 & -0.13871E-14 & -0.14010E-15 & -0.14011E-17 & -0.13885E-20 & -0.12623E-24 & -0.12484E-30 \\
0.12610E-09 & 0.14010E-09 & 0.14139E-10 & 0.14026E-12 & 0.12764E-15 & 0.13885E-20 & 0.12610E-26 \\
-0.12622E-05 & -0.14011E-05 & -0.14026E-06 & -0.12777E-08 & -0.14026E-12 & -0.14011E-17 & -0.12622E-23 \\
0.12610E-02 & 0.13885E-02 & 0.12764E-03 & 0.14026E-06 & 0.14139E-10 & 0.14010E-15 & 0.12610E-21 \\
-0.12484E+00 & -0.12623E+00 & -0.13885E-02 & -0.14011E-05 & -0.14010E-09 & -0.13871E-14 & -0.12484E-20 \\
0.11236E+01 & 0.12484E+00 & 0.12610E-02 & 0.12622E-05 & 0.12610E-09 & 0.12484E-14 & 0.11236E-20
\end{bmatrix},$$

$$V_M V_M^{-1} =$$
$$\begin{bmatrix}
0.10000E+01 & 0.00000E+00 & 0.21684E-18 & 0.21176E-21 & 0.00000E+00 & 0.19722E-30 & 0.00000E+00 \\
0.00000E+00 & 0.10000E+01 & -0.47705E-17 & -0.31764E-20 & -0.15510E-24 & -0.39443E-30 & -0.56424E-36 \\
0.22204E-15 & -0.10270E-14 & 0.10000E+01 & 0.15734E-18 & 0.46012E-23 & 0.33921E-28 & 0.89338E-34 \\
0.14655E-12 & -0.13486E-12 & -0.59344E-13 & 0.10000E+01 & -0.72381E-20 & -0.13531E-24 & -0.17467E-30 \\
-0.19217E-09 & 0.32179E-09 & 0.24745E-10 & 0.96021E-13 & 0.10000E+01 & 0.31471E-21 & 0.46481E-27 \\
0.20882E-05 & -0.38383E-06 & 0.43204E-06 & 0.26390E-08 & 0.15571E-12 & 0.10000E+01 & -0.69004E-23 \\
0.97025E-02 & -0.11445E+00 & -0.16108E-01 & -0.15871E-03 & 0.43217E-07 & -0.56032E-11 & 0.10000E+01
\end{bmatrix}.$$

As in Example 1, the significant improvement in accuracy in the computed inverse by incorporating both IM1 and IM2 is evident.

MODIFIED ALGORITHM WITH IM1 AND IM2 ($m = 5$, precision: real$\star$16).

$$V_M V_M^{-1} =$$
$$\begin{bmatrix}
0.10000E+01 & -0.15407E-32 & 0.12037E-34 & -0.11755E-37 & 0.00000E+00 & 0.10948E-46 & 0.00000E+00 \\
0.12326E-31 & 0.10000E+01 & -0.96296E-34 & 0.47020E-37 & 0.12914E-40 & -0.43791E-46 & -0.13573E-51 \\
-0.13559E-30 & -0.43449E-30 & 0.10000E+01 & 0.10109E-34 & -0.83943E-39 & 0.71379E-44 & 0.34767E-50 \\
0.11968E-28 & 0.33468E-28 & -0.17518E-29 & 0.10000E+01 & -0.26897E-36 & -0.74536E-41 & 0.21922E-47 \\
-0.72190E-26 & -0.11873E-25 & -0.13471E-26 & -0.19023E-28 & 0.10000E+01 & 0.60644E-37 & -0.74102E-44 \\
0.15974E-21 & 0.12399E-22 & 0.37006E-23 & 0.12929E-24 & 0.52997E-28 & 0.10000E+01 & -0.23543E-37 \\
-0.30388E-16 & -0.32086E-16 & -0.85046E-18 & -0.21101E-19 & -0.16852E-22 & 0.18880E-27 & 0.10000E+01
\end{bmatrix}.$$

EXAMPLE 3. Let $n = 7$, and let $\lambda_i = (-10)^{i-1}$, $i = 1, \ldots, 7$.

PARKER ALGORITHM.

$$V_M^{-1} =$$
$$\begin{bmatrix}
0.91744E+00 & 0.83404E-01 & -0.84247E-03 & -0.84154E-06 & 0.84247E-10 & 0.83404E-15 & -0.91744E-21 \\
0.83404E-01 & -0.84162E-01 & 0.75746E-03 & 0.76588E-06 & -0.76580E-10 & -0.75822E-15 & 0.83404E-21 \\
-0.84247E-03 & 0.75746E-03 & 0.84936E-04 & 0.76511E-07 & -0.77438E-11 & -0.76580E-16 & 0.84247E-22 \\
-0.84154E-06 & 0.76588E-06 & 0.76511E-07 & -0.84851E-09 & 0.76511E-13 & 0.76588E-18 & -0.84154E-24 \\
0.84247E-10 & -0.76580E-10 & -0.77438E-11 & 0.76511E-13 & 0.84936E-16 & 0.75746E-21 & -0.84247E-27 \\
0.83409E-15 & -0.75822E-15 & -0.76580E-16 & 0.76588E-18 & 0.75746E-21 & -0.84162E-25 & 0.83404E-31 \\
-0.85872E-21 & 0.83404E-21 & 0.84247E-22 & -0.84154E-24 & -0.84247E-27 & 0.83404E-31 & 0.91744E-36
\end{bmatrix},$$

$$V_M V_M^{-1} =$$
$$\begin{bmatrix}
0.10000E+01 & -0.16269E-17 & -0.59998E-19 & 0.52337E-22 & 0.64383E-26 & -0.17344E-32 & 0.64546E-37 \\
-0.58529E-14 & 0.10000E+01 & 0.12706E-17 & -0.73599E-21 & -0.11342E-24 & 0.53813E-30 & 0.78328E-36 \\
-0.53432E-12 & -0.62279E-14 & 0.10000E+01 & 0.36951E-19 & -0.10631E-22 & -0.86020E-29 & -0.19629E-34 \\
-0.53892E-10 & -0.43189E-12 & -0.48724E-14 & 0.10000E+01 & -0.11200E-19 & 0.10230E-25 & -0.40798E-31 \\
0.64000E+02 & -0.31528E-10 & 0.62290E-12 & 0.24647E-13 & 0.10000E+01 & -0.30215E-21 & -0.42283E-27 \\
0.58189E+08 & 0.64000E+02 & 0.85420E-07 & -0.10846E-08 & -0.14323E-11 & 0.10000E+01 & -0.13385E-21 \\
0.58776E+14 & 0.58189E+08 & 0.63995E+02 & -0.35584E-04 & -0.12800E-06 & 0.93081E-12 & 0.10000E+01
\end{bmatrix}.$$

MODIFIED ALGORITHM WITH IM1 ALONE ($m = 7$).

$$V_M^{-1} =$$
$$\begin{bmatrix}
0.91744E+00 & 0.83404E-01 & -0.84247E-03 & -0.84154E-06 & 0.84247E-10 & 0.83404E-15 & -0.91744E-21 \\
0.83404E-01 & -0.84162E-01 & 0.75746E-03 & 0.76588E-06 & -0.76580E-10 & -0.75822E-15 & 0.83404E-21 \\
-0.84247E-03 & 0.75746E-03 & 0.84936E-04 & 0.76511E-07 & -0.77438E-11 & -0.76580E-16 & 0.84247E-22 \\
-0.84154E-06 & 0.76588E-06 & 0.76511E-07 & -0.84851E-09 & 0.76511E-13 & 0.76588E-18 & -0.84154E-24 \\
0.84247E-10 & -0.76580E-10 & -0.77438E-11 & 0.76511E-13 & 0.84936E-16 & 0.75746E-21 & -0.84247E-27 \\
0.83404E-15 & -0.75822E-15 & -0.76580E-16 & 0.76588E-18 & 0.75746E-21 & -0.84162E-25 & 0.83404E-31 \\
-0.91744E-21 & 0.83404E-21 & 0.84247E-22 & -0.84154E-24 & -0.84247E-27 & 0.83404E-31 & 0.91744E-36
\end{bmatrix},$$

$V_M V_M^{-1} =$

$$\begin{bmatrix}
0.10000E+01 & -0.23246E-17 & -0.59945E-19 & 0.52337E-22 & 0.64383E-26 & -0.17344E-32 & 0.64546E-37 \\
0.97311E-16 & 0.10000E+01 & 0.12182E-17 & -0.73599E-21 & -0.11342E-24 & 0.53813E-30 & 0.78328E-36 \\
-0.17202E-15 & -0.73440E-15 & 0.10000E+01 & 0.36951E-19 & -0.10631E-22 & -0.86020E-29 & -0.19629E-34 \\
-0.76052E-13 & -0.57609E-13 & 0.20436E-14 & 0.10000E+01 & -0.11200E-19 & 0.10230E-25 & -0.40798E-31 \\
-0.70681E-10 & 0.28753E-10 & 0.11946E-11 & 0.24647E-13 & 0.10000E+01 & -0.30215E-21 & -0.42283E-27 \\
-0.22044E-05 & 0.80775E-06 & 0.10985E-06 & -0.10846E-08 & -0.14323E-11 & 0.10000E+01 & -0.13385E-21 \\
-0.38818E-01 & -0.91797E-01 & -0.21667E-02 & -0.35584E-04 & -0.12800E-06 & 0.93081E-12 & 0.10000E+01
\end{bmatrix}.$$

There is no further increase in accuracy (at least up to the first five decimal places) by using IM2. Once again, we see that the accuracy of the inverse is dramatically enhanced by the suggested improvements.

Modified Algorithm with IM1 and IM2 ($m = 5$, precision: real$\star$16).

$V_M V_M^{-1} =$

$$\begin{bmatrix}
0.10000E+01 & 0.76989E-33 & 0.35103E-35 & -0.74132E-39 & 0.12805E-42 & -0.53063E-47 & -0.31725E-53 \\
-0.57926E-32 & 0.10000E+01 & 0.27305E-34 & 0.12617E-37 & -0.17677E-40 & 0.13860E-46 & -0.35106E-52 \\
0.64977E-31 & 0.91748E-31 & 0.10000E+01 & -0.24514E-35 & 0.35407E-39 & 0.31968E-44 & 0.89723E-50 \\
0.14438E-29 & -0.21911E-29 & 0.51255E-30 & 0.10000E+01 & -0.22559E-36 & -0.66197E-41 & -0.32501E-47 \\
-0.92265E-26 & 0.51055E-26 & 0.20502E-26 & -0.11414E-28 & 0.10000E+01 & 0.48489E-37 & 0.11210E-43 \\
-0.46322E-22 & -0.39705E-22 & 0.66174E-23 & -0.71086E-25 & -0.75731E-28 & 0.10000E+01 & -0.58775E-38 \\
0.13878E-16 & -0.20817E-16 & -0.17347E-17 & 0.20329E-19 & 0.13235E-22 & -0.16156E-26 & 0.10000E+01
\end{bmatrix}.$$

Example 4. Let $n = 7$, and let $\lambda_1 = -3E5$, $\lambda_2 = 2E4$, $\lambda_3 = -1E2$, $\lambda_4 = 1$, $\lambda_5 = 1E2$, $\lambda_6 = 2E3$, and $\lambda_7 = 3E5$.

Parker Algorithm.

$V_M^{-1} =$

$$\begin{bmatrix}
-0.76664E-16 & 0.76692E-16 & -0.34750E-19 & -0.76671E-20 & 0.42415E-23 & -0.20568E-27 & 0.63874E-33 \\
-0.13952E-09 & 0.13959E-09 & -0.55808E-13 & -0.13959E-13 & 0.69760E-17 & 0.15510E-24 & -0.77511E-28 \\
0.46913E-02 & -0.47408E-02 & 0.49519E-04 & -0.25921E-07 & 0.11723E-11 & 0.28801E-18 & -0.13031E-22 \\
0.10007E+01 & -0.55036E-03 & -0.10004E-03 & 0.55036E-07 & -0.25005E-11 & -0.61151E-18 & 0.27796E-22 \\
-0.53430E-02 & 0.52925E-02 & 0.50521E-04 & -0.29255E-07 & 0.13352E-11 & 0.32505E-18 & -0.14842E-22 \\
0.13931E-05 & -0.13932E-05 & -0.69656E-10 & 0.13932E-09 & -0.69656E-14 & -0.15480E-20 & 0.77396E-25 \\
0.88759E-16 & -0.88824E-16 & 0.39655E-19 & 0.88804E-20 & -0.48533E-23 & 0.20566E-27 & 0.73980E-33
\end{bmatrix},$$

$V_M V_M^{-1} =$

$$\begin{bmatrix}
0.10000E+01 & 0.71851E-18 & -0.16905E-20 & 0.44462E-23 & 0.18485E-27 & -0.59135E-34 & -0.75281E-39 \\
-0.46199E-15 & 0.10000E+01 & -0.97111E-18 & -0.25886E-22 & -0.57620E-27 & -0.60050E-33 & 0.16657E-36 \\
-0.71018E-12 & 0.38842E-14 & 0.10000E+01 & 0.14398E-19 & 0.50884E-23 & 0.41949E-31 & -0.17591E-34 \\
-0.49817E-11 & -0.87233E-12 & -0.38784E-14 & 0.10000E+01 & 0.30316E-20 & 0.11722E-26 & 0.28617E-31 \\
-0.25600E+03 & -0.36783E-08 & 0.81471E-12 & 0.10896E-12 & 0.10000E+01 & 0.30885E-21 & 0.25591E-26 \\
-0.56323E+07 & -0.25600E+03 & 0.72323E-08 & 0.28685E-08 & 0.16760E-11 & 0.10000E+01 & 0.18136E-21 \\
-0.23154E+14 & -0.56323E+07 & -0.25599E+03 & 0.14782E-04 & -0.20582E-06 & 0.48260E-10 & 0.10000E+01
\end{bmatrix}.$$

Modified Algorithm with IM1 Alone ($m = 7$).

$V_M^{-1} =$

$$\begin{bmatrix}
-0.76649E-16 & 0.76692E-16 & -0.34750E-19 & -0.76671E-20 & 0.42415E-23 & -0.20568E-27 & 0.63874E-33 \\
-0.13952E-09 & 0.13959E-09 & -0.55808E-13 & -0.13959E-13 & 0.69760E-17 & 0.15510E-24 & -0.77511E-28 \\
0.46913E-02 & -0.47408E-02 & 0.49519E-04 & -0.25921E-07 & 0.11723E-11 & 0.28801E-18 & -0.13031E-22 \\
0.10007E+01 & -0.55036E-03 & -0.10004E-03 & 0.55036E-07 & -0.25005E-11 & -0.61151E-18 & 0.27796E-22 \\
-0.53430E-02 & 0.52925E-02 & 0.50521E-04 & -0.29255E-07 & 0.13352E-11 & 0.32505E-18 & -0.14842E-22 \\
0.13931E-05 & -0.13932E-05 & -0.69656E-10 & 0.13932E-09 & -0.69656E-14 & -0.15480E-20 & 0.77396E-25 \\
0.88776E-16 & -0.88824E-16 & 0.39655E-19 & 0.88804E-20 & -0.48533E-23 & 0.20566E-27 & 0.73980E-33
\end{bmatrix},$$

$V_M V_M^{-1} =$

$$\begin{bmatrix}
0.10000E+01 & 0.75811E-18 & -0.16707E-20 & 0.44462E-23 & 0.18485E-27 & -0.59135E-34 & -0.75281E-39 \\
-0.71110E-16 & 0.10000E+01 & -0.93149E-18 & -0.25886E-22 & -0.57620E-27 & -0.60050E-33 & 0.16657E-36 \\
0.28674E-14 & 0.39678E-14 & 0.10000E+01 & 0.14398E-19 & 0.50884E-23 & 0.41949E-31 & -0.17591E-34 \\
-0.12093E-11 & 0.48286E-13 & -0.36811E-14 & 0.10000E+01 & 0.30316E-20 & 0.11722E-26 & 0.28617E-31 \\
0.28716E-08 & -0.58167E-08 & 0.19864E-11 & 0.10896E-12 & 0.10000E+01 & 0.30885E-21 & 0.25591E-26 \\
0.70319E-04 & -0.21368E-04 & 0.20234E-07 & 0.28685E-08 & 0.16760E-11 & 0.10000E+01 & 0.18136E-21 \\
0.10445E+02 & -0.92969E+01 & 0.48504E-02 & 0.14782E-04 & -0.20582E-06 & 0.48260E-10 & 0.10000E+01
\end{bmatrix}.$$

No further improvement in accuracy is obtained by incorporating IM2 in addition to IM1; however, as in the previous example, the relative accuracy is significantly enhanced by the use of IM1. It may appear that there is large error in the inverse since the $(7,1)$ entry of $V_M V_M^{-1}$ is 10.445 instead of zero. However, as mentioned earlier, on comparing the results with those obtained using real⋆16 precision, it is observed that the inverses differ only in the 16^{th} significant digit.

MODIFIED ALGORITHM WITH IM1 AND IM2 ($m = 5$, precision: real⋆16).

$V_M V_M^{-1} =$

$$\begin{bmatrix} 0.10000\text{E}{+}01 & 0.31262\text{E}{-}34 & -0.65655\text{E}{-}36 & 0.70401\text{E}{-}39 & 0.12054\text{E}{-}43 & -0.18073\text{E}{-}50 & -0.63246\text{E}{-}56 \\ 0.15859\text{E}{-}31 & 0.10000\text{E}{+}01 & -0.34990\text{E}{-}34 & 0.43837\text{E}{-}37 & -0.13232\text{E}{-}42 & -0.21074\text{E}{-}48 & 0.17137\text{E}{-}52 \\ 0.29621\text{E}{-}30 & 0.75782\text{E}{-}30 & 0.10000\text{E}{+}01 & 0.18476\text{E}{-}35 & 0.24411\text{E}{-}40 & -0.83818\text{E}{-}46 & 0.11054\text{E}{-}50 \\ -0.54407\text{E}{-}28 & 0.30440\text{E}{-}27 & -0.47415\text{E}{-}30 & 0.10000\text{E}{+}01 & 0.94040\text{E}{-}37 & -0.44842\text{E}{-}43 & -0.30790\text{E}{-}47 \\ -0.47175\text{E}{-}24 & 0.73025\text{E}{-}24 & 0.94663\text{E}{-}29 & -0.36288\text{E}{-}28 & 0.10000\text{E}{+}01 & 0.00000\text{E}{+}00 & -0.44842\text{E}{-}43 \\ -0.50822\text{E}{-}20 & 0.11858\text{E}{-}19 & -0.82718\text{E}{-}24 & -0.62039\text{E}{-}24 & 0.20195\text{E}{-}27 & 0.10000\text{E}{+}01 & 0.00000\text{E}{+}00 \\ -0.88818\text{E}{-}15 & -0.17764\text{E}{-}14 & 0.43368\text{E}{-}18 & 0.10842\text{E}{-}18 & 0.00000\text{E}{+}00 & 0.00000\text{E}{+}00 & 0.10000\text{E}{+}01 \end{bmatrix}.$$

EXAMPLE 5. The purpose of this example is to show, that in some cases, no improvement might result with the use of IM1 and IM2. Let $n = 7$, and let $\lambda_1 = -4\text{E}5$, $\lambda_2 = 2\text{E}4$, $\lambda_3 = -1\text{E}2$, $\lambda_4 = 1$, $\lambda_5 = 1\text{E}2$, $\lambda_6 = -2\text{E}4$, and $\lambda_7 = 4\text{E}5$.

PARKER ALGORITHM.

$V_M^{-1} =$

$$\begin{bmatrix} 0.19580\text{E}{-}15 & -0.19580\text{E}{-}15 & -0.19091\text{E}{-}19 & 0.19581\text{E}{-}19 & -0.48903\text{E}{-}25 & -0.48951\text{E}{-}28 & 0.12238\text{E}{-}33 \\ 0.62661\text{E}{-}09 & -0.62658\text{E}{-}09 & -0.93992\text{E}{-}13 & 0.62658\text{E}{-}13 & 0.31331\text{E}{-}17 & -0.39161\text{E}{-}24 & -0.19582\text{E}{-}28 \\ 0.49506\text{E}{-}02 & -0.50001\text{E}{-}02 & 0.49506\text{E}{-}04 & 0.12532\text{E}{-}10 & -0.12407\text{E}{-}12 & -0.78127\text{E}{-}22 & 0.77353\text{E}{-}24 \\ 0.10001\text{E}{+}01 & 0.00000\text{E}{+}00 & -0.10001\text{E}{-}03 & 0.00000\text{E}{+}00 & 0.25065\text{E}{-}12 & 0.00000\text{E}{+}00 & -0.15627\text{E}{-}23 \\ -0.50506\text{E}{-}02 & 0.50001\text{E}{-}02 & 0.50506\text{E}{-}04 & -0.12532\text{E}{-}10 & -0.12658\text{E}{-}12 & 0.78127\text{E}{-}22 & 0.78916\text{E}{-}24 \\ -0.62655\text{E}{-}09 & 0.62658\text{E}{-}09 & 0.31328\text{E}{-}13 & -0.62658\text{E}{-}13 & 0.31328\text{E}{-}17 & 0.39161\text{E}{-}24 & -0.19580\text{E}{-}28 \\ -0.19580\text{E}{-}15 & 0.19580\text{E}{-}15 & 0.20070\text{E}{-}19 & -0.19581\text{E}{-}19 & -0.49001\text{E}{-}25 & 0.48951\text{E}{-}28 & 0.12238\text{E}{-}33 \end{bmatrix},$$

$V_M V_M^{-1} =$

$$\begin{bmatrix} 0.10000\text{E}{+}01 & -0.98853\text{E}{-}18 & -0.12902\text{E}{-}19 & 0.19953\text{E}{-}26 & 0.66637\text{E}{-}28 & -0.76477\text{E}{-}38 & -0.24427\text{E}{-}39 \\ -0.53953\text{E}{-}16 & 0.10000\text{E}{+}01 & -0.13408\text{E}{-}17 & 0.50011\text{E}{-}25 & 0.56940\text{E}{-}29 & 0.51857\text{E}{-}36 & -0.78543\text{E}{-}38 \\ 0.42811\text{E}{-}14 & -0.10980\text{E}{-}13 & 0.10000\text{E}{+}01 & -0.15676\text{E}{-}20 & 0.18136\text{E}{-}24 & -0.12145\text{E}{-}31 & -0.10059\text{E}{-}35 \\ -0.25132\text{E}{-}12 & -0.38299\text{E}{-}12 & -0.10725\text{E}{-}13 & 0.10000\text{E}{+}01 & -0.61571\text{E}{-}21 & 0.25046\text{E}{-}28 & 0.15948\text{E}{-}31 \\ 0.13425\text{E}{-}07 & 0.26776\text{E}{-}08 & -0.26271\text{E}{-}11 & -0.18580\text{E}{-}12 & 0.10000\text{E}{+}01 & -0.99468\text{E}{-}22 & -0.69420\text{E}{-}28 \\ -0.43488\text{E}{-}03 & 0.39220\text{E}{-}03 & -0.85274\text{E}{-}08 & -0.45984\text{E}{-}07 & 0.15510\text{E}{-}11 & 0.10000\text{E}{+}01 & -0.73412\text{E}{-}23 \\ 0.22525\text{E}{+}03 & -0.13494\text{E}{+}03 & -0.11566\text{E}{-}01 & 0.12993\text{E}{-}01 & -0.38810\text{E}{-}07 & -0.25139\text{E}{-}10 & 0.10000\text{E}{+}01 \end{bmatrix}.$$

Although there is a small error in the solution obtained using the Parker algorithm, no improvement in the solution results with our algorithm. However, as in the previous example, the error is only in the 16^{th} significant digit.

MODIFIED ALGORITHM WITH IM1 AND IM2 ($m = 5$, precision: real⋆16).

$V_M V_M^{-1} =$

$$\begin{bmatrix} 0.10000\text{E}{+}01 & 0.17296\text{E}{-}34 & 0.73443\text{E}{-}36 & 0.21192\text{E}{-}42 & -0.74116\text{E}{-}44 & 0.69489\text{E}{-}55 & 0.83503\text{E}{-}56 \\ -0.24437\text{E}{-}31 & 0.10000\text{E}{+}01 & -0.16461\text{E}{-}34 & 0.17795\text{E}{-}40 & -0.24384\text{E}{-}42 & 0.93638\text{E}{-}52 & -0.48563\text{E}{-}54 \\ -0.10975\text{E}{-}30 & -0.87558\text{E}{-}31 & 0.10000\text{E}{+}01 & -0.37974\text{E}{-}37 & 0.45512\text{E}{-}41 & 0.11974\text{E}{-}47 & 0.25122\text{E}{-}51 \\ 0.16467\text{E}{-}28 & -0.11853\text{E}{-}27 & -0.40984\text{E}{-}30 & 0.10000\text{E}{+}01 & 0.24681\text{E}{-}36 & -0.67262\text{E}{-}43 & 0.17106\text{E}{-}47 \\ 0.11374\text{E}{-}23 & 0.41359\text{E}{-}24 & 0.15777\text{E}{-}27 & -0.50487\text{E}{-}28 & 0.10000\text{E}{+}01 & 0.00000\text{E}{+}00 & -0.44842\text{E}{-}43 \\ -0.54210\text{E}{-}19 & 0.13553\text{E}{-}19 & 0.33087\text{E}{-}23 & 0.00000\text{E}{+}00 & 0.00000\text{E}{+}00 & 0.10000\text{E}{+}01 & 0.23510\text{E}{-}37 \\ 0.71054\text{E}{-}14 & 0.00000\text{E}{+}00 & -0.86736\text{E}{-}18 & 0.86736\text{E}{-}18 & 0.00000\text{E}{+}00 & 0.16156\text{E}{-}26 & 0.10000\text{E}{+}01 \end{bmatrix}.$$

4. CONCLUSIONS

Two changes have been proposed to the Parker algorithm which result in a considerably better relative accuracy in pathological situations. The relative accuracy is no worse than the Parker algorithm in all situations. The improvements seem to be the most significant when the λ_i are monotonically ordered. Since V_M^{-1} is computed accurately, the proposed algorithm can be used to get an accurate solution $\mathbf{x} = V_M^{-1}\mathbf{f}$ to the system of equations $V_M\mathbf{x} = \mathbf{f}$, as suggested in [5].

NOTE. The numerical implementation of this algorithm is available from the author upon request.

REFERENCES

1. F. Parker, Inverses of Vandermonde matrices, *Amer. Math. Monthly* **71**, 410–411, (1964).
2. H.J. Wertz, On the numerical inversion of a recurrent problem: The Vandermonde matrix, *IEEE Trans. on Automatic Control AC-10* **4**, 492, (1965).
3. J. Traub, Associated polynomials and uniform methods for the solution of linear problems, *SIAM Review* **8** (3), 277–301, (1966).
4. A. Bjorck and V. Pereyra, Solution of Vandermonde systems of linear equations, *Math. Computation* **24**, 893–903, (1970).
5. I. Gohberg and V. Olshevsky, The fast generalized Parker-Traub algorithm for inversion of Vandermonde and related matrices, *J. Complexity* **13** (2), 208–234, (1997).
6. C.S. Jog, On the explicit determination of the polar decomposition in n-dimensional vector spaces, *J. Elasticity* **66** (2), 159–169, (2002).
7. D. Kalman, A matrix proof of Newton's identities, `http://www.american.edu/academic.depts/cas/mathstat/ People/kalman/pdffiles/newtonsID.pdf`.